Computer Science & IT

Discrete and Engineering Mathematics

Comprehensive Theory

with **Solved Examples** and **Practice Questions**

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Discrete and Engineering Mathematics

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Discrete & Engineering Mathematics

Goal of the Subject

The mathematics of modern computer science is built almost entirely on discrete math, in particular Combinatorics and graph theory. Discrete math will help you with the "Algorithms, Complexity and Computability Theory" part of the focus more than programming language. The understanding of set theory, probability, matrices and combinations will allow us to analyze algorithms. We will be able to successfully identify parameters and limitations of your algorithms and have the ability to realize how complex a problem/solution is.

Discrete & Engineering Mathematics

INTRODUCTION

In this book we tried to keep the syllabus of Discrete & Engineering Mathematics around the GATE syllabus. Each topic required for GATE is crisply covered with illustrative examples and each chapter is provided with Student Assignment at the end of each chapter so that the students get a thorough revision of the topics that he/she had studied. This subject is carefully divided into eight chapters as described below.

Discrete Mathematics:

1. **Propositional Logic**: In this chapter we study logical connectives, well-Formed formulas, rules for inference, predicate calculus with Universal and Existential quantifiers.
2. **Combinatorics**: In this chapter we discuss the basic principles of counting, permutations, combinations, generating functions, binomial coefficients, summations and finally we discuss the recurrence relations.
3. **Set Theory and Algebra**: In this chapter we discuss the basic terms and definitions of set theory, Operations on sets, relations and types of relations , functions and their types and finally group theory, posets, lattices and boolean algebra.
4. **Graph Theory**: In this chapter we discuss the Special Graphs, isomorphism, vertex and edge connectivity, Euler graphs, Hamiltonian and planar graph, trees and enumeration of graphs.

Engineering Mathematics:

5. **Probability:** In this chapter we discuss the basic probability and axioms of probability, Basic concepts of statistics (mean, mode, variance and standard deviation), Discrete and continuous random variables and their distributions.
6. **Linear Algebra:** In this chapter we discuss the Special matrices, Algebra of matrices and their properties, inverse of a matrix, determinant of a matrix, solution of system of linear equations, LU decomposition method, Eigen values and Eigen vectors and finally we discuss the Cayley Hamilton theorem.
7. **Calculus**: In this chapter we discuss about Limits, continuity and differentiability, differentiation, partial derivatives, applications of differentiation (Mean value theorems, increasing and decreasing functions and maxima and minima of functions), methods of integration, and finally definite and indefinite integrals and their properties.

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CHAPTER 05

Probability

5.1 Some Fundamental Concepts

Sample Space and Event: Consider an experiment whose outcome is not predictable with certainty. Such an experiment is called a **random experiment**. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the **sample space** of experiment and is denoted by S.

Some examples follows:

(*i*) If the outcome of an experiment consist in the determination of the sex of a newborn child, then S = {*g*, *b*} where the outcome *g* means that the child is a girl and *b* is the boy.

(*ii*) If the outcome of an experiment consist of what comes up on a single dice, then S = {1, 2, 3, 4, 5, 6}

(*iii*) If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, 7; then S = {all 7! permutations of the (1, 2, 3, 4, 5, 6, 7)}

The outcome (2, 3, 1, 6, 5, 4, 7) means, for instances, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

Any subset E of the sample space is known as **Event**. That is, an event is a set consisting of some or all of the possible outcomes of the experiment.

If the outcome of the experiment is contained in E, then we say that E has occurred. Always $E \subseteq S$.

Since E and S are sets, theorems of set theory may be effectively used to represent and solve probability problems which are more complicated.

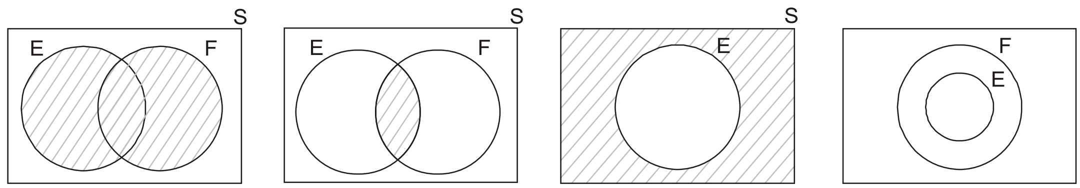
Examples: In the preceding example – (i) If $E_1 = \{g\}$, then E_1 is the event that the child is a girl.

Similarly, if $E_2 = \{b\}$.

Then E_2 is the event that the child is a boy. These are examples of simple events. Compounded events may consist of more than one outcome. Such as E = {1, 3, 5} for an experiment of throwing a dice. We say event E has happened if the dice comes up 1 or 3 or 5.

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consists of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if either E or F or both occurs. For instances, in dice example (i) if event $E_1 = \{1, 2\}$ and $E_2 = \{3, 4\}$, then $E \cup F = \{1, 2, 3, 4\}$.

That is $E \cup F$ would be another event consisting of 1 or 2 or 3 or 4. The event $E \cup F$ is called **union** of event E and the event F. Similarly, for any two events E and F we may also define the new event $E \cap F$, called **intersection** of E and F, to consists of all outcomes that are common to both E and F.



(a) Shaded region : $E \cup F$ (b) Shaded region : $E \cap F$ (c) Shaded region : E^C (d) $E \subset F$

Mutually Exclusive Events

Two events E and F are mutually exclusive, if $E \cap F = \phi$ i.e. $P(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).

Collectively Exhaustive Events

Two events E and F are collectively exhaustive, if $E \cup F = S$. i.e. together E and F include all possible outcomes, $P(E \cup F) = P(S) = 1$

DeMorgan's Law

(*i*) $\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$ (*ii*) $\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$

Example: $(E_1 \cup E_2)^C = E_1^C \cap E_2^C$ $(E_1 \cup E_2)^C = E_1^C \cup E_2^C$

Note that $E_1^C \cap E_2^C$ is called neither E_1 nor E_2. $E_1 \cup F_2$ is called either E_1 or E_2 (or both).

5.1.1 Approaches to Probability

There are 2 approaches to quantifying probability of an Event *E*.

1. **Classical Approach:** $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{|E|}{|S|}$

   i.e. the ratio of number of ways an event can happen to the number of ways sample space can happen, is the probability of the event. Classical approach assumes that all outcomes are equally likely.

2. **Frequency Approach:** Since sometimes all outcomes may not be equally likely, a more general approaches is the frequency approach, where probability is defined as the relative frequency of occurrence of *E*.

   $P(E) = \lim_{N \to \infty} \dfrac{n(E)}{N}$ where *N* is the number of times exp is performed & $n(E)$ is the no of times the event *E* occurs.

5.1.2 Axioms of Probability

Consider an experiment whose sample space is S. For each event E of the sample space S we assume that a number $P(E)$ is defined and satisfies the following three axioms.

Axiom-1: $0 \le P(E) \le 1$

Axiom-2: $P(S) = 1$

Axiom-3: For any sequence of mutually exclusive events $E_1, E_2, ...$(that is, events for which $E_i \cap E_j = \phi$ when $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Example: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ (E_1, E_2 are mutually exclusive)

Some Simple Propositions

It is to be noted that E and E° are always mutually exclusive and since E ∪ E° = S. We have by Axiom-(2) and (3) that : P(E ∪ E°) = P(E) + P(E°) = P(S) = 1

Proposition-1: P(E°) = 1 – P(E)

Proposition-2: If $E \subseteq F$, then $P(E) \le P(F)$

Proposition-3: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Prop - 3 is more general then axiom 3, since here E & F need not be mutually exclusive

Prop - 3 reduces to axiom - 3 when E, F mutually exclusive ($E \cap F = \phi$)

Prop - 3 may be extended for union of more sets as follows:

$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) + P(E \cap F \cap G)$

5.1.3 Conditional Probability

$$E/F = \frac{P(E \cap F)}{P(F)}$$

E/F is called the **conditional probability** of E given F.

Example - 5.1 A coin is flipped twice. What is the conditional probability that both flips result in heads, given that the first flip does?

Solution:

$$E/F = \frac{P(E \cap F)}{P(F)}$$

i.e. P (both are heads I first is heads)

$$= \frac{P(\text{bothheads \& firstishead})}{P(\text{firstishead})}$$

$$= \frac{P(\text{bothheads})}{P(\text{firsthead})} = \frac{1/4}{1/2} = \frac{1}{2}$$

5.1.4 The Multiplication Rule

$$P(E_1 \cap E_2) = P(E_1) * P(E_2/E_1) \quad ...(5.1)$$

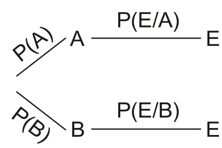
$$= P(E_2) * P(E_1/E_2) \quad ...(5.2)$$

Notice that (1) and (2) can be obtained from the following conditional probability formulas after cross multiplication.

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad \text{and} \quad P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

5.1.5 Rule of Total Probability and Bayes Theorem

Consider an event E which occurs via two different events A and B. Further more, Let A and B be mutually exclusive and collectively exhaustive events. This situation may be represented by following tree diagram



Now, the probability of E is given by value of total probability as:

$$P(E) = P(A \cap E) + P(B \cap E)$$
$$= P(A) * P(E/A) + P(B) *(E/B)$$

Sometimes we wish to know that, given that the event E has already occurred, what is the probability that it occurred with A?

i.e.
$$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A \cap E)}{P(A \cap E) + P(B \cap E)}$$
$$= \frac{P(A) * P(E|A)}{P(A) * P(E|A) + P(B) * P(E|B)}$$

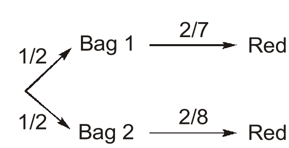
Notice that the denominator of Bayes theorem formula is obtained by using rule of total probability.

Example - 5.2 Suppose we have 2 bags. Bag 1 contains 2 red and 5 green marbles. Bag 2 contains 2 red and 6 green marbles. A person tosses a coin and if it is heads goes to bag 1 and draws a marble. If it is fails he goes to bag 2 and draws a marble. In this situation.

1. What is the probability that the marble drawn this is Red?
2. Given that the marble draw is red, what is probability that it came from bag 1.

Solution:

The tree diagram for above problem,



1. $\therefore$ $P(Red) = 1/2 \times 2/7 + 1/2 + 2/8$

2. $$P(bag1|Red) = \frac{P(bag1 \cap Red)}{P(Red)}$$

$$= \frac{\frac{1}{2} \times \frac{2}{7}}{\frac{1}{2} \times \frac{2}{7} \times \frac{1}{2} \times \frac{2}{8}} = \frac{\frac{1}{7}}{\frac{15}{56}} = 8/15$$

5.1.6 Independent Events

Two events are said to be **independent** if equation (A) holds.

$$P(E \cap F) = P(E) * P(F) \qquad ...(A)$$

Two events are said to be dependent if they are not independent.

Also it E and F are independent

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \times P(F)}{P(F)} = P(E)$$

Similarly, $P(E|F) = P(F)$

$P(E|F)$ is called **conditional probability** of E given F and P (E) in called **marginal probability** of E to distinguish it from $P(E|F)$.

P(F) is the marginal probability of F.

Example: A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then

$$P(E \cap F) = P(\text{Ace and Spade}) = \frac{1}{52}$$

$$P(E) = P(\text{Ace}) = \frac{4}{52} \text{ and } P(F) = P(\text{Spade}) = \frac{13}{52}$$

Here, $P(E \cap F) = P(F) * P(F)$

$\therefore$ E and F independent.

Proposition: If E and F are independent, then so are E and F^C, E^C & F, E^C & F^C.

Condition for three Events to be Independent: The events E, F ad G are said to be independent if P(EFG) = P(E) P(F) P(G)

$$\left.\begin{array}{l}\text{and } P(EF) = P(E)\,P(F) \\ \text{and } P(EG) = P(E)\,P(G) \\ \text{and } P(FG) = P(F)\,P(G)\end{array}\right. \left[\begin{array}{l}\text{E, F, G,} \\ \text{pairwise} \\ \text{independent}\end{array}\right]$$

It should be noted that if E, F and G are independent, then E will be independent of any event formed from F and G. For instance, E is independent of $F \cup G$.

5.2 Mean

Arithmetic Mean

The formula for calculating the arithmetic mean is: $\bar{x} = \dfrac{\Sigma x}{n}$

$\bar{x}$ -arithmetic mean

x-refers to the value of an observation

n-number of observations.

Example - 5.3 The number of visits made by ten mothers to a clinic were 8 6 5 5 7 4 5 9 7 4. Calculate the average number of visits.

Solution:

Σx = total of all these numbers of visits, that is the total number of visits made by all mothers.

$$8 + 6 + 5 + 5 + 7 + 4 + 5 + 9 + 7 + 4 = 60$$

$$\text{Number of mothers } n = 10$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{60}{10} = 6$$

The Arithmetic Mean of a Frequency Distribution

The formula for the arithmetic mean calculated from a frequency distribution has to be amended to include the frequency. It becomes:

$$\bar{x} = \frac{\Sigma(fx)}{\Sigma f}$$

Summary

- Two events E and F are mutually exclusive, if $E \cap F = f$ i.e. $P(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).
- **Axioms of Probability:**

  Axiom-1: $0 \le P(E) \le 1$

  Axiom-2: $P(S) = 1$

  Axiom-3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$.(that is, events for which $E_i \cap E_j = \phi$ when $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- **Median for Ungrouped Data:**

$$\text{Median} = \text{the } \frac{(n+1)}{2} \text{-th value}$$

  However, if n is even, we have two middle points

$$\text{Median} = \frac{\left(\frac{n}{2}\right)\text{-th value} + \left(\frac{n}{2}+1\right)\text{-th value}}{2}$$

- **Median for Grouped Data:**

$$\text{Median} = L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right] \times h$$

  Where,

  L = Lower limit of median class

  N = Total number of data items = ΣF

  F = Cumulative frequency of the class immediately preceding the median class

  f_m = Frequency of median class

  h = width of median class
- **Standard Deviation** is a measure of dispersion or variation amongst data. The positive square root of the variance is called the **'Standard Deviation'** of the given values.
- The probability of x success from n trials is given by $P(X = x) = nC_x\, p^x (1-p)^{n-x}$. Where p is the probability of success in any trial and $(1-p) = q$ is the probability of failure.
- **Uniform Distribution:**

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = E[X] = \frac{\beta+\alpha}{2}$$

$$\text{Variance} = V(X) = \frac{(\beta-\alpha)^2}{12}$$

- **Exponential Distribution:**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\text{Mean} = E[X] = \frac{1}{\lambda}$$

$$\text{Variance} = v(x) = 1/\lambda^2$$

- **Normal Distribution:**

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \; -\infty < x < \infty$$

$$\text{Mean} = E(X) = \mu$$

$$\text{Variance} = V(X) = \sigma^2$$

- **Standard Normal distribution:**

$$\text{Mean} = E(X) = 0$$

$$\text{Variance} = V(X) = 1$$

Hence the standard normal distribution is also referred to as the N(0, 1) distribution.

Student's Assignments

Q.1 Let P (E) denotes the probability of the event E. Given P(A) = 1, P(B) = $\frac{1}{2}$, then if A and B are independent, then the values of $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$ respectively are

(a) $\frac{1}{4}, \frac{1}{2}$ (b) $\frac{1}{2}, \frac{1}{4}$

(c) $\frac{1}{2}, 1$ (d) $1, \frac{1}{2}$

Q.2 If P(A) = $\frac{1}{2}$, P(A ∩ B) = $\frac{1}{4}$ then $P\left(\frac{B}{A}\right)$ =

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{3}{4}$ (d) 0

Q.3 A bag contains 5 black, 2 red, and 3 white marbles. Three marbles are drawn simultaneously. The probability that the drawn marbles are of the different color is

(a) $\frac{1}{6}$ (b) $\frac{1}{4}$

(c) $\frac{5}{6}$ (d) None of these

Q.4 A and B are equally likely and independent events. $p(A \cup B) = 0.1$. Then what is the value of $p(A)$?

(a) 0.032 (b) 0.046

(c) 0.513 (d) 0.05

Q.5 The probability of occurrence of an event. A is 0.7, the probability of non-occurrence of an event B is 0.45 and the probability of at least one of A and B not occurring is 0.6. The probability that at least one of A and B occurs is

(a) 0.4 (b) 0.6

(c) 1 (d) 0.85

Q.22 For the above distribution value of is

$$P\left(|x| \geq \frac{\sqrt{3}}{2}a\right)$$

(a) $\geq \frac{9}{4}$ (b) $\leq \frac{9}{5}$

(c) $\leq \frac{4}{9}$ (d) $\geq \frac{5}{9}$

Common Data Questions (23 and 24):

Analysis of the daily registration at an Examination on a certain day indicated that the source of registration from North India are 15%, South India are 35% and from western part of India are 50%. Further suppose that the probabilities that a registration being a free registration from these parts are 0.01, 0.05, and 0.02, respectively.

Q.23 Find the probability that a registration chosen at random is a free registration

(a) 0.603   (b) 0.029

(c) 0.009   (d) None of these

Q.24 Find the probability that a randomly chosen registration comes from south India, given that it is a free registration.

(a) 60%   (b) 3%

(c) 17 %   (d) None of these

Q.25 A manufacturer produces IC chips, 1% of which are defective. Find the probability that in a box containing 100 chips, no defective are found. Use Poisson distribution approximation to binomial distribution?

(a) 0.366   (b) 0.368

(c) 0.1   (d) None of these

Answer Key:

1. (d)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (d)
11. (b)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (d)
16. (c)	**17.** (d)	**18.** (d)	**19.** (b)	**20.** (a)
21. (b)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (b)

Student's Assignments Explanations

1. (d)

Since A and B independent events

$p(A|B) = p(A) = 1$ and $p(B|A) = p(B) = \frac{1}{2}$.

2. (b)

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

5. (d)

Given,

$$p(A) = 0.7$$
$$p(\bar{B}) = 0.45$$
$$p(\bar{A} \cup \bar{B}) = 0.6$$
$$p(A \cup B) = ?$$
$$p(B) = 1 - p(\bar{B})$$
$$= 1 - 4.5 = 0.55$$
$$p(A \cap B) = 1 - p(\overline{A \cap B})$$
$$= 1 - p(\bar{A} \cup \bar{B})$$
$$= 1 - 0.6 = 0.4$$

Now, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
$= 0.7 + 0.55 - 0.4 = 0.85$

$\therefore$ $p(A \cup B) = 0.85$

6. (a)

If X is a random variable, then

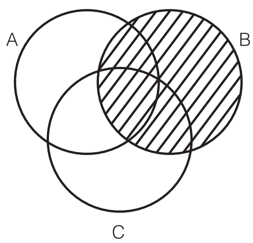
$$\Sigma p(X) = 1$$
$$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$
$$\Rightarrow \quad k = \frac{1}{49}$$

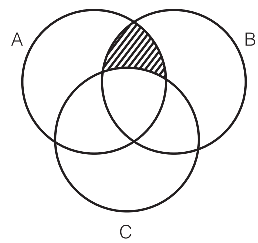
7. (a)

$P(3 < x \leq 6) = 9k + 11k + 13k = 33k$

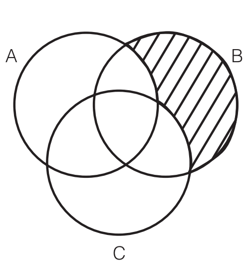
$$\therefore \quad P(3 < x \leq 6) = \frac{33}{49}$$

8. (a)

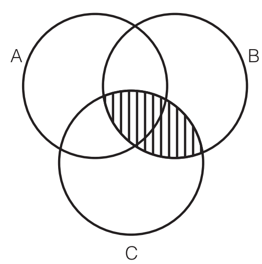




$$P(B)\ P(A \cap B \cap \bar{C}) = \frac{1}{3}$$



$P(\bar{A} \cap B \cap \bar{C})$



$P(B \cap C)$

From the above Venn diagram

$$P(B \cap C)\ P(B) - P(A \cap B \cap \bar{C}) - P(\bar{A} \cap B \cap \bar{C})$$

$$= \frac{3}{4} - \frac{1}{3} - \frac{1}{3} = \frac{1}{12}$$

9. (a)

Let X be the random variable that represents the sum of 2 tickets.

The probability distribution table of X is

X	3	4	5	6	7	8	9	10	11
$p(X)$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{15}$

$$E(X) = \Sigma X p(X)$$

$$= 3 \times \frac{1}{15} + 4 \times \frac{1}{15} + 5 \times \frac{2}{15} + \ldots$$

$$= \frac{105}{15} = 7$$

10. (d)

The possible combinations for at least one dice being 6 is given by 11 ordered pairs below:

(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)

$\therefore$ Probability that at least one dice is 6 = $\frac{11}{36}$

Alternatively we can solve this problem by another method:

p(6 on I dice or 6 on II dice)

= 1 – p(not 6 on I dice and not 6 on II dice)

$$= 1 - \frac{5}{6} \times \frac{5}{6} = 1 - \frac{25}{36} = \frac{11}{36}$$

11. (b)

For $f(x)$ to be a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \quad \int_0^2 kx\,dx = 1$$

$$\Rightarrow \quad k\left[\frac{x^2}{2}\right]_0^2 = 1 \Rightarrow 2k = 1$$

$$\Rightarrow \quad k = \frac{1}{2}$$

$$p(1 \le x \le 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{1}{2} x\,dx = \frac{3}{4}$$

12. (b)

There are 5 faces that are heads out of a total of 8, so the probability is $\frac{5}{8}$. Let A be the event that the upper face is a head, and B be the event that the lower face is heads.

$$\Pr[A] = \Pr[B] = \frac{5}{8}$$

$$\Pr[A \cap B] = \frac{2}{4} = \frac{1}{2}$$

So, $$\Pr[B \mid A] = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{2}}{\frac{5}{8}} = \frac{4}{5}$$

13. (d)

$$3X + 12 \equiv 0 \pmod{33}$$

$$\Rightarrow \quad 3X \equiv -12 \pmod{33}$$