# Tangent Lines and Linear Approximations

Students should be able to:

- Determine the slope of tangent line to a curve at a point
- Determine the equations of tangent lines
- Approximate a value on a function using a tangent line and determine if the estimate is an over- or under-approximation based on concavity of the function

**Point-Slope Form of the Equation of the Tangent Line**

$$y - y_0 = m(x - x_0)$$

Replacing $(x_0, y_0)$ with $(c,\ f(c))$ :

$$y = f(c) + f'(c)(x - c)$$

which leads to the form for linear approximation:

$$f(x) \approx f(c) + f'(c)(x - c)$$

Students need to be able to recognize different ways that a tangent line approximation can appear on the AP exam:

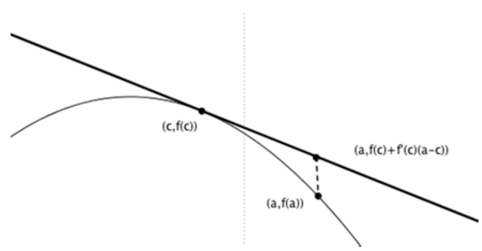
- Tangent line approximation
- Linear approximation
- Linearization
- Euler's method (BC)

It is also important to know if the linear approximation is an over- or under-approximation:

over-approximation

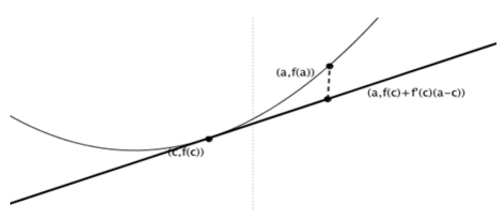
$f(c) + f'(c)(a - c) > f(a)$ if the graph of $y = f(x)$ is concave downward near $x = c$.

This will occur if $f''(x) < 0$ near $x = c$.




under-approximation

$f(c) + f'(c)(a - c) < f(a)$ if the graph of $y = f(x)$ is concave upward near $x = c$. This will occur if $f''(x) > 0$ near $x = c$.




Unlike a Riemann Sum, determining whether a tangent line is an over/under approximation is not related to whether a function is increasing or decreasing. When determining (or justifying) whether a tangent line is an over or under approximation, the concavity of the function must be discussed. It is important to look at the concavity on the interval from the point of tangency to the x-value of the approximation, not just the concavity at the point of tangency.

*Example Justification*: The approximation of $f(1.1)$ using the tangent line of f(x) at the point $x = 1$ is an over-approximation of the function because $f''(x) < 0$ on the interval 1 < x < 1.1.

Multiple Choice

1. (calculator not allowed)
If the line tangent to the graph of the function $f$ at the point $(1,\ 7)$ passes through the point $(-2,\ -2)$, then $f'(1)$ is

(A) $-5$
(B) 1
(C) 3
(D) 7
(E) undefined

2. (calculator not allowed)

An equation of the line tangent to the graph of $y = \cos(2x)$ at $x = \frac{\pi}{4}$ is

(A) $y - 1 = -\left(x - \frac{\pi}{4}\right)$

(B) $y = 2\left(x - \frac{\pi}{4}\right)$

(C) $y = -2\left(x - \frac{\pi}{4}\right)$

(D) $y - 1 = -2\left(x - \frac{\pi}{4}\right)$

(E) $y = -\left(x - \frac{\pi}{4}\right)$

3. (calculator not allowed)
Let $f$ be a differentiable function with $f(2) = 3$ and $f'(2) = -5$, and let $g$ be the function $g(x) = x \cdot f(x)$. Which of the following is an equation of the line tangent to the graph of $g$ at the point where $x = 2$?

(A) $y = 3x$
(B) $y - 6 = -5(x - 2)$
(C) $y - 6 = -10(x - 2)$
(D) $y - 3 = -5(x - 2)$
(E) $y - 6 = -7(x - 2)$

4. (calculator not allowed)
   An equation of the line tangent to the graph of $f(x)=x(1-2x)^3$ at the point $(1,\ -1)$ is

   (A) $y=-7x+6$
   (B) $y=-6x+5$
   (C) $y=-2x+1$
   (D) $y=2x-3$
   (E) $y=7x-8$

5. (calculator not allowed)
   An equation of the line tangent to the graph of $y=\dfrac{2x+3}{3x-2}$ at the point $(1,\ 5)$ is

   (A) $13x-y=8$
   (B) $13x+y=18$
   (C) $x-13y=64$
   (D) $x+13y=66$
   (E) $-2x+3y=13$

6. (calculator not allowed)
   Let $f$ be a differentiable function such that $f(3)=2$ and $f'(3)=5$. If the tangent line at $x=3$ is used to find an approximation to a zero of $f$, that approximation is

   (A) 0.4
   (B) 0.5
   (C) 2.6
   (D) 3.4
   (E) 5.5

7. (calculator not allowed)

At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent line parallel to the line $2x - 4y = 3$?

(A) $\left(\frac{1}{2}, -\frac{1}{2}\right)$

(B) $\left(\frac{1}{2}, \frac{1}{8}\right)$

(C) $\left(1, -\frac{1}{4}\right)$

(D) $\left(1, \frac{1}{2}\right)$

(E) $(2, 2)$

8. (calculator not allowed)

The approximate value of $y = \sqrt{4 + \sin(x)}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$ is

(A) 2.00
(B) 2.03
(C) 2.06
(D) 2.12
(E) 2.24

9. (calculator not allowed)

What is the slope of the tangent to the curve $3y^2 - 2x^2 = 6 - 2xy$ at the point $(3, 2)$?

(A) 0

(B) $\frac{4}{9}$

(C) $\frac{7}{9}$

(D) $\frac{6}{7}$

(E) $\frac{5}{3}$

10. (calculator not allowed) appropriate for AB

The slope of the line tangent to the graph of $\ln(xy) = x$ at the point where $x = 1$ is

(A) 0
(B) 1
(C) $e$
(D) $e^2$
(E) $1 - e$

11. (calculator allowed)

Let $f$ be the function given by $f(x) = 2e^{4x^2}$. For what value of $x$ is the slope of the line tangent to the graph of $f$ at $(x, f(x))$ equal to 3?

(A) 0.168
(B) 0.276
(C) 0.318
(D) 0.342
(E) 0.551

12. (calculator allowed)

Which of the following is an equation of the line tangent to the graph of $f(x) = x^4 + 2x^2$ at the point where $f'(x) = 1$?

(A) $y = 8x - 5$
(B) $y = x + 7$
(C) $y = x + 0.763$
(D) $y = x - 0.122$
(E) $y = x - 2.146$

13. (calculator allowed)

Let $f$ be the function given by $f(x) = 3e^{2x}$ and let $g$ be the function given by $g(x) = 6x^3$. At what value of $x$ do the graphs of $f$ and $g$ have parallel tangent lines?

(A) $-0.701$
(B) $-0.567$
(C) $-0.391$
(D) $-0.302$
(E) $-0.258$

Free Response

14. (calculator not allowed)

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3(1+3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

15. (calculators not allowed)

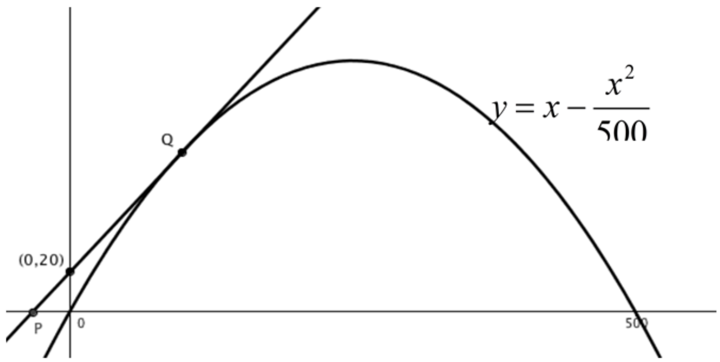
| $x$ | $-1.5$ | $-1.0$ | $-0.5$ | 0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1$ | $-4$ | $-6$ | $-7$ | $-6$ | $-4$ | $-1$ |
| $f'(x)$ | $-7$ | $-5$ | $-3$ | 0 | 3 | 5 | 7 |

Let $f$ be a function that is differentiable for all real numbers. The table above gives the values of $f$ and its derivative $f'$ for selected points $x$ in the closed interval $-1.5 \le x \le 1.5$. The second derivative of $f$ has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(b) Write an equation of the line tangent to the graph of $f$ at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

16. (calculator not allowed)

Line $l$ is tangent to the graph of $y = x - \frac{x^2}{500}$ at the point $Q$, as shown in the figure below.




(a) Find the $x$-coordinate of point $Q$.

(b) Write an equation for line $l$.

(c) Suppose the graph of $y = x - \frac{x^2}{500}$ shown in the figure, where $x$ and $y$ are measured in feet, represents a hill. There is a 50-foot tree growing vertically at the top of the hill. Does a spotlight at point $P$ directed along line $l$ shine on any part of the tree? Show the work that leads to your conclusion.

17. (calculator not allowed)

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

(b) Find all points $(x,\ y)$ on the curve where the line tangent to the curve has slope $\dfrac{1}{2}$.

(c) Show that there are no points $(x,\ y)$ on the curve where the line tangent to the curve is horizontal.